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REEB FOLIATIONS ON S^5 AND CONTACT 5-MANIFOLDS VIOLATING THE THURSTON-BENNEQUIN INEQUALITY

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Abstract. We obtain the following two results through foliation theoretic approaches including a review of Lawson's construction of a codimension-one foliation on the 5-sphere:
1) The standard contact structure on the 5-sphere deforms to 'Reeb foliations'.
2) We define a 5-dimensional Lutz tube which contains a plastikstufe. Inserting it into any contact 5-manifold, we obtain a contact structure which violates the Thurston-Bennequin inequality for a convex hypersurface with contact-type boundary.

1. Introduction and preliminaries

The first aim of this paper is to show that the standard contact structure $\mathcal{D}_0$ on S^5 deforms via contact structures into spinnable foliations, which we call Reeb foliations (§2). Here a spinnable foliation is a codimension-one foliation associated to an open-book decomposition whose binding is fibred over S^1. In 1971, Lawson[16] constructed a spinnable foliation on S^5 associated to a Milnor fibration. We construct such a spinnable foliation on S^5 as the limit $\mathcal{D}_1$ of a family $\{\mathcal{D}_t\}_{0\leq t<1}$ of contact structures. Since S^5 is compact, the family $\{\mathcal{D}_t\}_{0\leq t<1}$ can be traced by a family of diffeomorphisms $\varphi_t : S^5 \to S^5$ wth $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_*\mathcal{D}_0 = \mathcal{D}_t$ (Gray's stability).

The second aim is to show that any contact 5-manifold admits a contact structure which violates the Thurston-Bennequin inequality for a *convex* hypersurface (§3). We define a 5-*dimensional Lutz tube* and explain how to insert it into a given contact 5-manifold to violate the inequality. Moreover a 5-dimensional Lutz tube contains a *plastikstufe*, which is an obstruction to symplectic fillability found by Niederkrüger[22] and Chekanov. A different *Lutz twist* on a contact manifold (M^{2n+1}, α) was recently introduced in Etnyre-Pancholi[6] as a modification of the contact structure $\mathcal{D} = \ker\alpha$ near an n-dimensional submanifold. Contrastingly, the core of our Lutz tube is a codimension-two contact submanifold. We change the standard contact structure on S^5 by inserting a Lutz tube along the binding of the open-book decomposition of a certain Reeb foliation.

The author[21] also showed that any contact manifold of dimension>3 violates the Thurston-Bennequin inequality for a *non-convex* hypersurface. However he conjectures that the inequality holds for any *convex* hypersurface in the standard S^{2n+1}. See §4 for related problems.

The rest of this section is the preliminaries.

1.1. **Thurston-Bennequin inequality.** A *positive* (resp. *negative*) *contact manifold* consists of an oriented $(2n+1)$-manifold M^{2n+1} and a 1-form α on M^{2n+1} with $\alpha\wedge(d\alpha)^n > 0$ (resp. $\alpha\wedge(d\alpha)^n < 0$). The (co-)oriented hyperplane distribution $\mathcal{D} = \ker\alpha$ is called the *contact structure* on the contact manifold (M^{2n+1}, α). In the case where (M^{2n+1}, α) is positive, the symplectic structure $d\alpha|\ker\alpha$ on the oriented vector bundle $\ker\alpha$ is also positive, i.e., $(d\alpha)^n|\ker\alpha > 0$. Hereafter we assume that all contact structures and symplectic structures are positive.

In this subsection, we assume that any compact connected oriented hypersurface Σ embedded in a contact manifold (M^{2n+1}, α) tangents to the contact structure $\ker\alpha$ at finite number of interior points. Note that the hyperplane field $\ker\alpha$ is maximally non-integrable. Let $S_+(\Sigma)$ (resp. $S_-(\Sigma)$) denote the set of the positive (resp. negative) tangent points, and $S(\Sigma)$ the union $S_+(\Sigma)\cup S_-(\Sigma)$. The sign of the tangency at $p \in S(\Sigma)$ coincides with the sign of $\{(d\alpha|\Sigma)^n\}_p$. Considering on $(\ker\alpha, d\alpha|\ker\alpha)$, we see that the symplectic orthogonal of the intersection $T\Sigma\cap\ker\alpha$ forms an oriented line field L on Σ, where the singularity of L coincides with $S(\Sigma)$.

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Definition 1.1. The singular oriented foliation $\mathcal{F}_\Sigma$ defined by $T\mathcal{F}_\Sigma = L$ is called the *characteristic foliation* on Σ with respect to the contact structure $\ker\alpha$.

Put $\beta = \alpha|\Sigma$ and take any volume form ν on Σ. Then we see that the vector field X on Σ defined by $\iota_X\nu = \beta\wedge(d\beta)^{n-1}$ is a positive section of L. Moreover,

$$\iota_X\{\beta\wedge(d\beta)^{n-1}\} = -\beta\wedge\iota_X(d\beta)^{n-1} = 0, \quad \beta\wedge(d\beta)^{n-1}\neq 0 \quad\Longrightarrow\quad \beta\wedge\iota_X d\beta = 0.$$

Thus the flow generated by X preserves the conformal class of β. Since ν is arbitrary, we may take X as any positive section of L. Therefore the 1-form β defines a holonomy invariant transverse contact structure of the characteristic foliation $\mathcal{F}_\Sigma$.

On the other hand, for any volume form $\mu(\neq\nu)$ on Σ, we see that the sign of $\operatorname{div} X = (\mathcal{L}_X\mu)/\mu$ at each singular point $p\in S(\Sigma)$ coincides with the sign of p. Thus $\mathcal{F}_\Sigma$ contains the information about the sign of the tangency to the contact structure $\ker\alpha$. We also define the *index* $\operatorname{Ind} p = \operatorname{Ind}_X p$ of a singular point $p\in S(\Sigma)$ by using the above vector field X.

Definition 1.2. Suppose that the boundary $\partial\Sigma$ of the above hypersurface Σ is non-empty, and the characteristic foliation $\mathcal{F}_\Sigma$ is positively (i.e., outward) transverse to $\partial\Sigma$. Then we say that Σ is a hypersurface with *contact-type* boundary. Note that $\beta|\partial\Sigma = \alpha|\partial\Sigma$ is a contact form.

Remark. The *Liouville vector field* X on a given exact symplectic manifold $(\Sigma, d\lambda)$ with respect to a primitive 1-form λ of $d\lambda$ is defined by $\iota_X d\lambda = \lambda$. If X is positively transverse to the boundary $\partial\Sigma$, then $(\partial\Sigma, \lambda|\partial\Sigma)$ is called the contact-type boundary. The above definition is a natural shift of this notion into the case of hypersurfaces in contact manifolds.

Let D^2 be an embedded disk with contact-type boundary in a contact 3-manifold. We say that D^2 is *overtwisted* if the singularity $S(D^2)$ consists of a single sink point. Note that a sink point is a negative singular point since it has negative divergence. A contact 3-manifold is said to be *overtwisted*, or *tight* depending on whether there exists an overtwisted disk with contact-type boundary in it, or not. We can show that the existence of an overtwisted disk with contact-type boundary is equivalent to the existence of an *overtwisted disk with Legendrian boundary*, which is an embedded disk D' similar to the above D^2 except that the characteristic foliation $\mathcal{F}_{D'}$ tangents to the boundary $\partial D'$, where $\partial D'$ or $-\partial D'$ is a closed leaf of $\mathcal{F}_{D'}$.

Let Σ be *any* surface with contact-type (i.e., transverse) boundary embedded in the standard S^3. Then Bennequin[1] proved the following inequality which implies the tightness of S^3:

Thurston-Bennequin inequality. $\displaystyle\sum_{p\in S_-(\Sigma)} \operatorname{Ind} p \leq 0.$

Eliashberg proved the same inequality for symplectically fillable contact 3-manifolds ([3]), and finally for all tight contact 3-manifolds ([4]). Recently Niederkrüger[22] and Chekanov found a $(n+1)$-dimensional analogue of an overtwisted disk with Legendrian boundary — a *plastikstufe* which is roughly the trace $K^{n-1}\times D^2$ of an overtwisted disk D^2 with Legendrian boundary travelling along a closed integral submanifold $K^{n-1}\subset M^{2n+1}$. However, in order to create some meaning of the above inequality in higher dimensions, we need a $2n$-dimensional analogue of an overtwisted disk with contact-type boundary.

Remark. The Thurston-Bennequin inequality can also be written in terms of relative Euler number: The vector field $X\in T\Sigma\cap\ker\alpha$ is a section of $\ker\alpha|\Sigma$ which is canonical near the boundary $\partial\Sigma$. Thus under a suitable boundary condition we have

$$\langle e(\ker\alpha),\, [\Sigma,\partial\Sigma]\rangle = \sum_{p\in S_+(\Sigma)} \operatorname{Ind} p - \sum_{p\in S_-(\Sigma)} \operatorname{Ind} p.$$

Then the Thurston-Bennequin inequality can be expressed as

$$-\langle e(\ker\alpha),\, [\Sigma,\partial\Sigma]\rangle \leq -\chi(\Sigma).$$

There is also an absolute version of the Thurston-Bennequin inequality for a closed hypersurface Σ with $\chi(\Sigma)\leq 0$, which is expressed as $|\langle e(\ker\alpha),\, [\Sigma]\rangle| \leq -\chi(\Sigma)$, or equivalently

$$\sum_{p\in S_-(\Sigma)} \operatorname{Ind} p \leq 0 \qquad\text{and}\qquad \sum_{p\in S_+(\Sigma)} \operatorname{Ind} p \leq 0.$$

The absolute version trivially holds if the Euler class $e(\ker\alpha) \in H^{2n}(M;\mathbb{Z})$ is a torsion. Note that the inequality and its absolute version can be defined for any oriented plane field on an oriented 3-manifold M^3 (see Eliashberg-Thurston[5]). They are originally proved for a foliation on M^3 without Reeb components by Thurston (see [24]).

1.2. **Convex hypersurfaces.** In this subsection we explain Giroux's convex hypersurface theory outlined in [9] and add a possible relative version to it.

A vector field Y on a contact manifold (M,α) which satisfies $\alpha\wedge\mathcal{L}_Y\alpha = 0$ is called a *contact vector field*. Let V_α denote the set of all contact vector fields on (M,α). It is well-known that the linear map $\alpha(\cdot): V_\alpha \to C^\infty(M)$ is an isomorphism.

Definition 1.3. 1) For a given contact vector field Y on a contact manifold (M,α), the function $H = \alpha(Y) \in C^\infty(M)$ is called the *Hamiltonian function* of Y. Conversely for a given function $H \in C^\infty(M)$, the unique contact vector field Y with $\alpha(Y) = H$ is called the *contact Hamiltonian vector field* of H. The contact Hamiltonian vector field of 1 lies in the degenerate direction of $d\alpha$ and is called the *Reeb field* of α.

2) A closed oriented hypersurface Σ embedded in a contact manifold (M,α) is said to be *convex* if there exists a contact vector field transverse to Σ.

Let Y be a contact vector field positively transverse to a closed convex hypersurface Σ, and $\Sigma\times(-\varepsilon,\varepsilon)$ a neighbourhood of $\Sigma = \Sigma\times\{0\}$ with $Y = \partial/\partial z$ $(z\in(-\varepsilon,\varepsilon))$. We may assume that the contact form α is Y-invariant after rescaling it by multiplying a suitable positive function. Note that this rescaling does not change the level set $\{\alpha(Y) = 0\}$. By perturbing Y in V_α if necessary, we can modify the Hamiltonian function $H = \alpha(Y)$ so that the level set $\{H = 0\}$ is a regular hypersurface of the form $\Gamma\times(-\varepsilon,\varepsilon)$ in the above neighbourhood $\Sigma\times(-\varepsilon,\varepsilon)$, where $\Gamma\subset M$ is a codimension-2 submanifold. Put $h = H|\Sigma$.

Definition 1.4. The submanifold $\Gamma = \{h = 0\} \subset \Sigma$ is called the *dividing set* on Σ with respect to Y. Γ divides Σ into the *positive region* $\Sigma_+ = \{h \geq 0\}$ and the *negative region* $\Sigma_- = -\{h \leq 0\}$ so that $\Sigma = \Sigma_+ \cup (-\Sigma_-)$. We orient Γ as $\Gamma = \partial\Sigma_+ = \partial\Sigma_-$.

Note that $\pm Y|\{\pm H > 0\}$ is the Reeb field of $\alpha/|H| = \beta/|H| \pm dz$, where β is the pull-back of $\alpha|\Sigma$ under the projection along Y. Since the $2n$-form

$$\Omega = (d\beta)^{n-1}\wedge(Hd\beta + n\beta dH)$$

satisfies $\Omega\wedge dz = \alpha\wedge(d\alpha)^n > 0$, the characteristic foliation $\mathcal{F}_\Sigma$ is positively transverse to the dividing set Γ. Thus Γ is a positive contact submanifold of (M,α). The open set $U = \{|H| < \varepsilon'\}$ is of the form $(-\varepsilon',\varepsilon')\times\Gamma\times(-\varepsilon,\varepsilon)$ for sufficiently small $\varepsilon' > 0$. Let $\rho(H) > 0$ be an even function of H which is increasing on $H > 0$, and coincides with $1/|H|$ except on $(-\varepsilon',\varepsilon')$. Then we see that $d(\rho\alpha)|\mathrm{int}\,\Sigma_\pm$ are symplectic forms.

On the other hand, let $(\Sigma_\pm, d\lambda_\pm)$ be compact exact symplectic manifolds with the same contact-type boundary $(\partial\Sigma_\pm,\mu)$, where we fix the primitive 1-forms $\lambda_\pm$ and assume that $\mu = \lambda_\pm|\partial\Sigma_\pm$. Then $\lambda_i + dz$ is a z-invariant contact form on $\Sigma_i\times\mathbb{R}$ ($i = +$ or $-$, $z\in\mathbb{R}$).

Definition 1.5. The contact manifold $(\Sigma_i\times\mathbb{R}, \lambda_i + dz)$ is called the *contactization* of $(\Sigma_i, d\lambda_i)$. Take a collar neighbourhood $(-\varepsilon', 0]\times\partial\Sigma_i \subset \Sigma_i$ such that

$$\lambda_i + dz|((-\varepsilon',0]\times\partial\Sigma_i\times\mathbb{R}) = e^s\mu + dz \quad (s\in(-\varepsilon',0]).$$

We modify $\lambda_i + dz$ near $(-\varepsilon',0]\times\partial\Sigma_i\times\mathbb{R}$ in a canonical way into a contact form α_i with

$$\alpha_i|((-\varepsilon',0]\times\partial\Sigma_i\times\mathbb{R}) = e^{-s^2/\varepsilon'}\mu - \frac{s}{\varepsilon'}dz.$$

We call the contact manifold $(\Sigma_i\times\mathbb{R}, \alpha_i)$ the *modified contactization* of $(\Sigma_i, d\lambda_i)$.

Remark. The above symplectic manifold $(\Sigma_i, d\lambda_i)$ can be fully extended by attaching the half-symplectization $(\mathbb{R}_{\geq 0}\times\partial\Sigma_i, d(e^s\mu))$ to the boundary. The interior of the modified contactization is then contactomorphic to the contactization of the fully extended symplectic manifold.

The modified contactizations $\Sigma_+ \times \mathbb{R}$ and $\Sigma_- \times \mathbb{R}'$ match up to each other to form a connected contact manifold $((\Sigma_+ \cup (-\Sigma_-)) \times \mathbb{R}, \alpha)$ where $\mathbb{R}' = -\mathbb{R}$. Indeed, α can be written near $\Gamma \times \mathbb{R} = \partial\Sigma_+ \times \mathbb{R} = \partial\Sigma_- \times (-\mathbb{R}')$ as

$$\alpha|(-\varepsilon', \varepsilon') \times \Gamma \times \mathbb{R} = e^{-s^2/\varepsilon'}\mu - \frac{s}{\varepsilon'}dz \quad (s \in (-\varepsilon', \varepsilon'),\, z \in \mathbb{R}).$$

Definition 1.6. The contact manifold $((\Sigma_+ \cup (-\Sigma_-)) \times \mathbb{R}, \alpha)$ is called the *unified contactization* of $\Sigma = \Sigma_+ \cup (-\Sigma_-)$.

Since $(-\Sigma)_+ = \Sigma_-$ and $(-\Sigma)_- = \Sigma_+$, the unified contactization of $-\Sigma = \Sigma_- \cup (-\Sigma_+)$ can be obtained by turning the unified contactization of $\Sigma_+ \cup (-\Sigma_-)$ upside-down. Note that $-Y \in V_\alpha$. Clearly, a small neighbourhood of any convex hypersurface $\Sigma_+ \cup (-\Sigma_-)$ is contactomorphic to a neighbourhood of $(\Sigma_+ \cup (-\Sigma_-)) \times \{0\}$ in the unified contactization.

Conceptually, a convex hypersurface in contact topology play the same role as a contact-type hypersurface in symplectic topology — both are powerful tools for cut-and-paste because they have canonical neighbourhoods modeled on the unified contactization and the symplectization. Further Giroux[9] showed that any closed surface in a contact 3-manifold is smoothly approximated by a convex one. This fact closely relates contact topology with differential topology in this dimension. On the other hand, there exists a hypersurface which cannot be smoothly approximated by a convex one if the dimension of the contact manifold is greater than three (see [21]).

Definition 1.7. A compact connected oriented embedded hypersurface Σ with non-empty contact-type boundary in a contact manifold (M, α) is said to be *convex* if there exists a contact vector field Y such that $\alpha(Y)|\partial\Sigma > 0$ and Y is transverse to Σ.

Put $h = \alpha(Y)|\Sigma$ after perturbing Y. Then the dividing set $\Gamma = \{h = 0\}$ divides Σ into the positive region $\Sigma_+ = \{h \geq 0\}$ and the (possibly empty) negative region $\Sigma_- = -\{h \leq 0\}$ so that

$$\Sigma = \Sigma_+ \cup (-\Sigma_-) \quad \text{and} \quad \partial\Sigma = \partial\Sigma_+ \setminus \partial\Sigma_- \neq \emptyset.$$

Note that the above definition avoids touching of Γ to the contact-type boundary $\partial\Sigma$. Now the Thurston-Bennequin inequality can be written as

Thurston-Bennequin inequality for convex hypersurfaces. $\chi(\Sigma_-) \leq 0 \quad (\mathit{or} \quad \Sigma_- = \emptyset)$.

Suppose that there exists a convex disk $\Sigma = D^2$ with contact-type boundary in a contact 3-manifold which is the union $\Sigma_+ \cup (-\Sigma_-)$ of a negative disk region Σ_- and a positive annular region Σ_+. Then the convex disk Σ violates the Thurston-Bennequin inequality and is called a *convex overtwisted disk* ($\chi(\Sigma_-) = 1 > 0$). Conversely, it is clear that any overtwisted disk with contact-type boundary is also approximated by a convex overtwisted disk.

Definition 1.8. A *convex overtwisted hypersurface* is a connected convex hypersurface $\Sigma_+ \cup (-\Sigma_-)$ with contact-type boundary which satisfies $\chi(\Sigma_-) > 0$.

Note that any convex overtwisted hypersurface Σ contains a connected component of Σ_+ whose boundary is disconnected. This relates to Calabi's question on the existence of a compact connected exact symplectic $2n$-manifold ($n > 1$) with disconnected contact-type boundary. McDuff[18] found the first example of such a manifold. Here is another example:

Example 1.9. (Mitsumatsu[19], Ghys[8] and Geiges[7]) To obtain a symplectic 4-manifold with disconnected contact-type boundary, we consider the mapping torus $T_A = T^2 \times [0,1]/A \ni ((x,y),z)$ of a linear map $A \in SL(2;\mathbb{Z})$ ($A : T^2 \times \{1\} \to T^2 \times \{0\}$) with $\operatorname{tr} A > 2$. Let $dvol_{T^2}$ be the standard volume form on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $v_\pm$ eigenvectors of A which satisfy

$$Av_\pm = a^{\pm 1} v_\pm, \quad \text{where} \quad a > 1 \quad \text{and} \quad dvol_{T^2}(v_+, v_-) > 0.$$

In general, a cylinder $[-1,1] \times M^3$ admits a symplectic structure with contact-type boundary if M^3 admits a co-orientable Anosov foliation (Mitsumatsu[19]). In the case where $M^3 = T_A$, the 1-forms $\beta_\pm = \pm a^{\mp z} dvol_{T^2}(v_\pm, \cdot)$ define Anosov foliations. Then the cylinder

$$(W_A = [-1,1] \times T_A, d(\beta_+ + s\beta_-)) \quad (s \in [-1,1])$$

is a symplectic manifold with contact-type boundary $(-T_A) \sqcup T_A$.

Using the above cylinder W_A, we construct a convex overtwisted hypersurface in §3.

2. Convergence of contact structures to foliations

First we define a supporting open-book decomposition on a closed contact manifold.

Definition 2.1. Let (M^{2n+1}, α) be a closed contact manifold and $\mathcal{O}$ an open-book decomposition on M^{2n+1} by pages P_θ ($\theta \in \mathbb{R}/2\pi\mathbb{Z}$). Suppose that the binding $(N^{2n-1} = \partial P_\theta, \alpha|N^{2n-1})$ of $\mathcal{O}$ is a contact submanifold. Then if there exists a positive function ρ on M^{2n+1} such that

$$d\theta \wedge \{d(\rho\alpha)\}^n > 0 \quad \text{on} \quad M^{2n+1} \setminus N^{2n-1},$$

the open-book decomposition $\mathcal{O}$ is called a *supporting open-book decomposition* on (M^{2n+1}, α).

The function ρ can be taken so that $\rho\alpha$ is axisymmetric near the binding. Precisely, we can modify the function ρ near a tubular neighbourhood $N^{2n-1} \times D^2$ except on the binding $N^{2n-1} \times \{0\}$, if necessary, so that with respect to the polar coordinates (r, θ) on the unit disk D^2

i) the restriction $\rho\alpha|(N^{2n-1} \times D^2)$ is of the form $f(r)\mu + g(r)d\theta$,
ii) μ is the pull-back $\pi^*(\rho\alpha|N^{2n-1})$ under the projection $\pi : N^{2n-1} \times D^2 \to N^{2n-1}$,
iii) $f(r)$ is a positive function of r on $N^{2n-1} \times D^2$ with $f'(r) < 0$ on $(0, 1]$, and
iv) $g(r)$ is a weakly increasing function with $g(r) \equiv r^2$ near $r = 0$ and $g(r) \equiv 1$ near $r = 1$.

Next we prove the following theorem.

Theorem 2.2. *Let $\mathcal{O}$ be a supporting open-book decomposition on a closed contact manifold (M^{2n+1}, α) of dimension greater than three $(n > 1)$. Suppose that the binding N^{2n-1} of $\mathcal{O}$ admits a non-zero closed 1-form ν with $\nu \wedge \{d(\rho\alpha|N^{2n-1})\}^{n-1} \equiv 0$ where ρ is a function on M^{2n+1} satisfying all of the above conditions. Then there exists a family of contact forms $\{\alpha_t\}_{0\le t<1}$ on M^{2n+1} which starts with $\alpha_0 = \rho\alpha$ and converges to a non-zero 1-form α_1 with $\alpha_1 \wedge d\alpha_1 \equiv 0$. That is, the contact structure $\ker\alpha$ then deforms to a spinnable foliation.*

Proof. Take smooth functions $f_1(r)$, $g_1(r)$, $h(r)$ and $e(r)$ of $r \in [0, 1]$ such that

i) $f_1 \equiv 1$ near $r = 0$, $f_1 \equiv 0$ on $[1/2, 1]$, $f_1' \le 0$ on $[0, 1]$,
ii) $g_1 \equiv 1$ near $r = 1$, $g_1 \equiv 0$ on $[0, 1/2]$, $g_1' \ge 0$ on $[0, 1]$,
iii) $h \equiv 1$ on $[0, 1/2]$, $h \equiv 0$ near $r = 1$,
iv) e is supported near $r = 1/2$, and $e(1/2) \ne 0$.

Put $f_t(r) = (1 - t)f(r) + tf_1(r)$, $g_t(r) = (1 - t)g(r) + tg_1(r)$ and

$$\alpha_t|(N \times D^2) = f_t(r)\{(1 - t)\mu + th(r)\nu\} + g_t(r)d\theta + te(r)dr,$$

where ν also denotes the pull-back $\pi^*\nu$. We extend α_t by

$$\alpha_t|(M \setminus (N \times D^2)) = \tau\rho\alpha + (1 - \tau)d\theta \quad \text{where} \quad \tau = (1 - t)^2.$$

Then we see from $d\nu \equiv 0$ and $\nu \wedge (d\mu)^{n-1} \equiv 0$ that $\alpha_t \wedge (d\alpha_t)^n$ can be written as

$$nf_t^{n-1}(1 - t)^n(g_t' f_t - f_t' g_t)\mu \wedge (d\mu)^{n-1} \wedge dr \wedge d\theta \quad \text{on} \quad N \times D^2 \quad \text{and}$$

$$\tau^{n+1}\rho^{n+1}\alpha \wedge (d\alpha)^n + \tau^n(1 - \tau)d\theta \wedge \{d(\rho\alpha)\}^n \quad \text{on} \quad M \setminus (N \times D^2).$$

Therefore we have

$$\alpha_t \wedge (d\alpha_t)^n > 0 \quad (0 \le t < 1), \quad \alpha_1 \wedge d\alpha_1 \equiv 0 \quad \text{and} \quad \alpha_1 \ne 0.$$

This completes the proof of Theorem 2.2. □

Remark. 1) A similar result in the case where $n = 1$ is contained in the author's paper[20]: Any contact structure $\ker\alpha$ on a closed 3-manifold deforms to a spinnable foliation.
2) The orientation of the compact leaf $\{r = 1/2\}$ depends on the choice of the sign of the value $e(1/2)$. Here the choice is arbitrary.

We give some examples of the above limit foliations which relate to the following proposition on certain T^2-bundles over the circle.

Proposition 2.3 (Van Horn[26]). *Let $T_{A_{m,0}}$ denote the mapping torus $T^2\times[0,1]/A_{m,0}\ni((x,y),z)$ of the linear map $A_{m,0}=\begin{pmatrix}1&0\\m&1\end{pmatrix}:T^2\times\{1\}\to T^2\times\{0\}$ $(m\in\mathbb{N})$. Then $\ker(dy+mzdx)$ is the unique Stein fillable contact structure on $T_{A_{m,0}}$ (up to contactomorphism). Moreover it admits a supporting open-book decomposition $\mathcal{O}_{m,0}$ such that*

i) *the page is a m-times punctured torus, and*
ii) *the monodromy is the right-handed Dehn twist along (the disjoint union of m loops parallel to) the boundary of the page.*

Let $\mathbb{C}^3$ be the $\xi\eta\zeta$-space, and π_ξ,π_η and π_ζ denote the projections to the axes.

Example 2.4 (Lawson's foliation). The link L of the singular point $(0,0,0)$ of the complex surface $\{\xi^3+\eta^3+\zeta^3=0\}\subset\mathbb{C}^3$ is diffeomorphic to the T^2-bundle $T_{A_{3,0}}$. (To see this, consider the projective curve $\{\xi^3+\eta^3+\zeta^3=0\}\subset\mathbb{C}P^2$ diffeomorphic to T^2. Since L is the union of the Hopf fibres over this torus, it is also a T^2-bundle over the circle.) Moreover, since L is Stein fillable, it is contactomorphic to $(T_{A_{3,0}},\mu=dy+3zdx)$. Indeed the open-book decomposition $\mathcal{O}_{3,0}$ in Proposition 2.3 is equivalent to the supporting open-book decomposition $\{\arg(\pi_\xi|L)=\theta\}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}$ on L. (To see this, regard ξ as a parameter and consider the curve $C_\xi=\{\eta^3+\zeta^3=-\xi^3\}$ on the $\eta\zeta$-plane, which is diffeomorphic to $T^2\setminus\{\text{three points}\}$. Then we can see that the fibration $\{(\{\xi\}\times C_\xi)\cap B^6\}_{|\xi|=\varepsilon}$ $(0<\varepsilon\ll 1)$ is equivalent to the page fibration of $\mathcal{O}_{3,0}$, where B^6 denotes the unit hyperball of $\mathbb{C}^3\approx\mathbb{R}^6\ni(x_1,y_1,\dots,x_3,y_3)$.) We put

$$\Lambda=\sum_{i=1}^{3}(x_idy_i-y_idx_i)\quad\text{and}\quad V_{\varepsilon,\theta}=\{\xi^3+\eta^3+\zeta^3=\varepsilon e^{\sqrt{-1}\theta}\}\cap B^6\quad(\theta\in\mathbb{R}/2\pi\mathbb{Z}).$$

Then Gray's stability implies that $\partial V_{\varepsilon,\theta}\subset(S^5,\Lambda|S^5)$ is contactomorphic to L. Since $\xi^3+\eta^3+\zeta^3$ is a homogeneous polynomial, the 1-form $\Lambda|V_{\varepsilon,\theta}$ is conformal to the pull-back of the restriction of Λ to $\Sigma_{\varepsilon,\theta}=\{\rho p\,|\,\rho>0,p\in V_{\varepsilon,\theta}\}\cap S^5\subset\{\arg(\xi^3+\eta^3+\zeta^3)=\theta\}$ under the central projection. Indeed $\rho x_id(\rho y_i)-\rho y_id(\rho x_i)=\rho^2(x_idy_i-y_idx_i)$ holds for any function ρ. Thus the fibration $\{\Sigma_{\varepsilon,\theta}\}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}$ extends to a supporting open-book decomposition on the standard S^5. We put $\nu=dz$ and apply Theorem 2.2 to obtain a limit foliation $\mathcal{F}_{3,0}$ of the standard contact structure. This is the memorable first foliation on S^5 discovered by Lawson[16].

For other examples, we need the following lemma essentially due to Giroux and Mohsen.

Lemma 2.5 (see [11]). *Let $f:\mathbb{C}^n\to\mathbb{C}$ be a holomorphic function with $f(0,\dots,0)=0$ such that the origin $(0,\dots,0)$ is an isolated critical point. Take a sufficiently small hyperball $B_\varepsilon=\{|z_1|^2+\dots+|z_n|^2=\varepsilon^2\}$. Then there exists a supporting open-book decomposition on the standard S^{2n-1} such that the binding is contactomorphic to the link $\{f=0\}\cap\partial B_\varepsilon$ and the page fibration is equivalent to the fibration $\{\{f=\delta\}\cap B_\varepsilon\}_{|\delta|=\varepsilon'}$ $(0<\varepsilon'\ll\varepsilon)$.*

Proof. Take the hyperball $B'_\varepsilon=\{|z_1|^2+\dots+|z_{n+1}|^2=\varepsilon^2\}$ on $\mathbb{C}^{n+1}$ and consider the complex hypersurface $\Sigma_k=\{z_{n+1}=kf(z_1,\dots,z_n)\}\cap B'_\varepsilon$ with contact-type boundary $\partial\Sigma_k$ $(k\geq 0)$. Then Gray's stability implies that $\partial\Sigma_k$ is contactomorphic to $\partial\Sigma_0(=\partial B_\varepsilon)$. From $dz_{n+1}|\Sigma_\infty=0$ and $(x_{n+1}dy_{n+1}-y_{n+1}dx_{n+1})(-y_{n+1}\partial/\partial x_{i+1}+x_{n+1}\partial/\partial y_{n+1})\geq 0$, we see that $\{\arg(f|\partial\Sigma_k)=\theta\}_{\theta\in S^1}$ is a supporting open-book decomposition of $\partial\Sigma_k$ equivalent to $\{\{f=\delta\}\cap B_\varepsilon\}_{|\delta|=\varepsilon'}$ if k is sufficiently large and $\varepsilon'>0$ is sufficiently small. □

Example 2.6. Consider the polynomials $f_1=\xi^6+\eta^3+\zeta^2$ and $f_2=\xi^4+\eta^4+\zeta^2$. Then the link L_m of the singular point $(0,0,0)\in\{f_m=0\}$ is contactomorphic to the above T^2-bundle $T_{A_{m,0}}$ with the contact form $\mu=dy+mzdx$ $(m=1,2)$. Indeed $\mathcal{O}_{m,0}$ is equivalent to the supporting open-book decomposition $\{\arg(\pi_\xi|L_m)=\theta\}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}}$ on L_m. (To see this, regard ξ as a parameter and consider $C_\xi=\{f_m=0\}$ on the $\eta\zeta$-plane, which is diffeomorphic to $T^2\setminus\{m\,\text{points}\}$. Then we can see that the fibration $\{(\{\xi\}\times C_\xi)\cap B^6\}_{|\xi|=\varepsilon}$ is equivalent to the page fibration of $\mathcal{O}_{m,0}$.) On the other hand, Lemma 2.5 sais that there exists a supporting open-book decomposition on the standard S^5 which is equivalent to the Milnor fibration with binding L_m. We put $\nu=dz$ and apply Theorem 2.2 to obtain a limit foliation $\mathcal{F}_{m,0}$ $(m=1,2)$.

Definition 2.7.

1) Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with $f(0,\dots,0) = 0$ such that the origin is an isolated critical point or a regular point of f. If the origin is singular, the Milnor fibre has the homotopy type of a bouquet of n-spheres. Suppose that the Euler characteristic of the Milnor fibre is positive, that is, the origin is regular if n is odd. Then we say that the Milnor fibration is *PE* (=positive Euler characteristic).
2) Let $\mathcal{O}$ be a supporting open-book decomposition of the standard S^{2n+1}. Suppose that the binding is the total space of a fibre bundle π over $\mathbb{R}/\mathbb{Z} \ni t$, and the Euler characteristic of the page is positive. Then if $\nu = \pi^* dt$ satisfies the assumption of Theorem 2.2, the resultant limit foliation is called a *Reeb foliation*.

The above $\mathcal{F}_{m,0}$ $(m = 1, 2, 3)$ are Reeb foliations associated to PE Milnor fibrations. To obtain other examples of foliations associated to more general Milnor fibrations, Grauert's topological characterization of Milnor fillable 3-manifolds is instructive ([12], see also [2]).

3. Five-dimensional Lutz tubes

In this section, we define a 5-dimensional Lutz tube by means of an open-book decomposition whose page is a convex hypersurface. We insert the Lutz tube along the binding of a certain supporting open-book decomposition on the standard S^5. Then we obtain a new contact structure on S^5 which violates the Thurston-Bennequin inequality for a convex hypersurface. We also show that the 5-dimensional Lutz tube contains a plastikstufe.

3.1. **Convex open-book decompositions.** We explain how to construct a contact manifold with an open-book decomposition by convex pages.

Proposition 3.1. *Let $(\Sigma_\pm, d\lambda_\pm)$ be two compact exact symplectic manifolds with contact-type boundary. Suppose that there exists an inclusion $\iota : \partial\Sigma_- \to \partial\Sigma_+$ such that $\iota^*(\lambda_+|\partial\Sigma_+) = \lambda_-|\partial\Sigma_-$. Let φ be a self-diffeomorphism of the union $\Sigma = \Sigma_+ \cup_\iota (-\Sigma_-)$ supported in $\operatorname{int}\Sigma_+ \sqcup \operatorname{int}(-\Sigma_-)$ which satisfies*

$$(\varphi|\Sigma_\pm)^*\lambda_\pm - \lambda_\pm = dh_\pm$$

for suitable positive functions $h_\pm$ on $\Sigma_\pm$. We choose some connected components of $\partial\Sigma_+ \setminus \iota(\partial\Sigma_-)$ and take their disjoint union B. Then there exists a smooth map Φ from the unified contactization $\Sigma \times \mathbb{R}$ to a compact contact manifold (M, α) such that

i) *$\Phi|(\Sigma \setminus B) \times \mathbb{R}$ is a cyclic covering which is locally a contactomorphism,*
ii) *$P_0 = \Phi(\Sigma \times \{0\}) \approx \Sigma$ is a convex page of an open-book decomposition $\mathcal{O}$ on (M, α),*
iii) *$\Phi(B \times \mathbb{R}) \approx B$ is the binding contact submanifold of $\mathcal{O}$, and*
iv) *φ is the monodromy map of $\mathcal{O}$.*

Proof. In the case where $\Sigma_- = \emptyset$ and $B = \partial\Sigma_+$, this proposition was proved in Giroux[10] (essentially in Thurston-Winkelnkemper[25]). Then $\mathcal{O}$ is a supporting open-book decomposition. In general, let $\Sigma \times \mathbb{R}$ be the unified contactization, i.e., the union of the modified contactizations of $\Sigma_\pm$ by the attaching map $(\iota, -\mathrm{id}_{\mathbb{R}'}) : \partial\Sigma_- \times \mathbb{R}' \to \partial\Sigma_+ \times \mathbb{R}(= -\mathbb{R}')$. Consider the quotient $\Sigma \times (\mathbb{R}/2\pi c\mathbb{Z})$ $(c > 0)$, and cap-off the boundary components $B \times (\mathbb{R}/2\pi c\mathbb{Z})$ by replacing the collar neighbourhood $(-\varepsilon', 0] \times B \times (\mathbb{R}/2\pi c\mathbb{Z})$ with $(B \times D^2, (\lambda_+|B) + r^2 d\theta)$ where $\theta = z/c$. Adding constants to $h_\pm$ if necessary, we may assume that $h_\pm$ are the restrictions of the same function h. We change the identification $(x, z + 2\pi c) \sim (x, z)$ to $(x, z + h) \sim (\varphi(x), z)$ before capping-off the boundary $B \times S^1$. This defines the map Φ and completes the proof. □

Remark. Giroux and Mohsen proved that any symplectomorphism supported in $\operatorname{int}\Sigma_+$ is isotopic via such symplectomorphisms to φ with $\varphi^*\lambda_+ - \lambda_+ = dh_+$ $(\exists h_+ > 0)$. They also proved that there exists a supporting open-book decomposition on any closed contact manifold by interpreting the result of Ibort-Martinez-Presas[14] on the applicability of Donaldson-Auroux's asymptotically holomorphic methods to complex functions on contact manifolds (see [10]).

3.2. **Definition of Lutz tubes.** Let W_A be the symplectic manifold with disconnected contact-type boundary $(-T_A)\sqcup T_A$ in Example 1.9. Then, by a result of Van Horn[26], each of the boundary component is a Stein fillable contact manifold. Note that $\mathrm{tr}(A^{-1}) > 2$ and $(-T_A) \approx T_{A^{-1}}$. Precisely, there exists a supporting open-book decomposition on T_A described as follows.

Proposition 3.2. 1) (Honda[13], see also Van Horn[26]) *Any element $A \in SL(2;\mathbb{Z})$ with $\mathrm{tr}\, A > 2$ is conjugate to at least one of the elements*

$$A_{m,k} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & k_m \\ 0 & 1 \end{pmatrix} \in SL(2;\mathbb{Z}),$$

where $m \in \mathbb{Z}_{>0}$, $k = (k_1, \dots, k_m) \in (\mathbb{Z}_{\geq 0})^m$ and $k_1 + \cdots + k_m > 0$.

2) (Van Horn[26]) *The contact manifold $(T_{A_{m,k}}, (\beta_- + \beta_+)|T_{A_{m,k}})$ $(k \neq 0)$ admits a supporting open-book decomposition which is determined up to equivalence by the following data:*

Page: *The page P is the m-times punctured torus $\bigcup_{i\in\mathbb{Z}_m} P_i$, where P is divided into three-times punctured spheres P_i by mutually disjoint loops γ_i with $P_i \cup P_{i+1} = \gamma_i$.*

Monodromy: *The monodromy is the composition $\tau(\partial P) \circ \prod_{i=1}^{m} \{\tau(\gamma_i)\}^{k_i}$, where $\tau(\gamma)$ denotes the right-handed Dehn twist along γ.*

These data determines a PALF (=positive allowable Lefschetz fibration) structure of the canonical Stein filling V of the contact manifold $T_{A_{m,k}}$ if $m \geq 2$ (see Loi-Piergallini [17] and Giroux [10]). Here we see through the PALF structure that attaching 1-handles to the page corresponds to attaching 1-handles to the canonical Stein filling, and adding right-handed Dehn twists along non-null-homologous loops to the monodromy corresponds to attaching 2-handles to the canonical Stein filling. Thus we have

$$\chi(V) = 1_{(=\#\{\text{0-handle}\})} - (m+1)_{(=\#\{\text{1-handles}\})} + (m + k_1 + \cdots + k_m)_{(=\#\{\text{2-handles}\})} > 0.$$

In the case where $m = 1$, let ℓ_1 and ℓ_2 denote simple loops on P which intersect transversely at a single point. It is well-known that $\tau(\partial P)$ is isotopic to $(\tau(\ell_1) \circ \tau(\ell_2))^6$. This new expression determines a PALF structure with $12+k_1$ singular fibres on the canonical Stein filling V of $T_{A_{1,(k_1)}}$. Then we have $\chi(V) = 1 - 2 + 12 + k_1 > 11$. Thus the following corollary holds.

Corollary. *The contact manifold $(T_A, (\beta_- + \beta_+)|T_A)$ admits a Stein filling V with $\chi(V) > 0$.*

Then the unified contactization of $W_A \cup (-V)$ under the natural identification $\{1\} \times T_A \sim \partial V$ contains a convex overtwisted hypersurface $(W_A \cup (-V)) \times \{0\}$.

Now we define a 5-dimensional Lutz tube.

Definition 3.3. Putting $\Sigma_+ = W_A$, $\Sigma_- = \emptyset$, $B = -(\{-1\} \times T_A) \approx T_{A^{-1}}$ and $\varphi = \mathrm{id}_\Sigma$, we apply Proposition 3.1 to obtain a contact manifold $T_{A^{-1}} \times D' \approx T_A \times D^2$ $(D' = -D^2)$, which we call the 5-*dimensional Lutz tube* associated to A $(\mathrm{tr}\, A > 2)$.

The next proposition explains how to insert a Lutz tube.

Proposition 3.4. *Let $(V, d\lambda)$ be an exact strong symplectic filling of T_A with* $\mathrm{tr}\, A > 2$, *$\psi : V \to V$ a diffeomorphism supported in* $\mathrm{int}\, V$. *Suppose that $\chi(V) > 0$ and $\psi^*\lambda - \lambda = dh$ $(\exists h > 0)$. Putting $\Sigma_+ = V$, $\Sigma_- = \emptyset$, $B = \partial V$ and $\varphi = \psi$, we apply* Proposition 3.1 *to obtain a closed contact manifold (M^5, α) with an open-book decomposition whose binding is T_A. Next we consider the union $W_A \cup (-V)$ with respect to the natural identification $\iota : \partial V \to \{1\} \times T_A \subset W_A$. Again putting $\Sigma_+ = W_A$, $\Sigma_- = V$, $B = \partial W_A \setminus \iota(\partial V)$ and $\varphi =$ (the trivial extension of ψ^{-1}), we apply* Proposition 3.1 *to obtain another contact manifold (M', α') with an open-book decomposition, where the page is a convex overtwisted hypersurface and the binding is $T_{A^{-1}}$. Then there exists a diffeomorphism from M^5 to M' which preserves the orientation and sends T_A to $-T_{A^{-1}}$. We may consider that (M', α') is obtained by inserting the 5-dimensional Lutz tube associated to A along the binding T_A of a supporting open-book decomposition on (M^5, α).*

Remark. 1) Similarly, we can insert a 5-dimensional Lutz tube along *any* codimension-2 contact submanifold with trivial normal bundle which is contactomorphic to T_A $(\mathrm{tr} A > 2)$.

Particularly, we can insert the Lutz tube associated to A^{-1} along the core of the Lutz tube associated to A. Then we obtain a 5-dimensional analogue of the full Lutz tube.

2) We may consider the original 3-dimensional (half) Lutz tube as a trivial open-book decomposition by positive annuli whose binding is a connected component of the boundary. That is, starting with the exact symplectic annulus $([-1,1]\times S^1, sd\theta)$ $(s\in[-1,1],\theta\in S^1)$ we can construct the 3-dimensional Lutz tube by Proposition 3.1 ($\varphi=\mathrm{id}$).

3) Geiges[7] constructed an exact symplectic 6-manifold $[-1,1]\times M^5$ with contact-type boundary, where M^5 is a certain T^4-bundle over the circle. From his example, we can also construct a 7-dimensional Lutz tube. The author suspects that this Lutz tube enables us to change not only the contact structure but also the homotopy class of the almost contact structure of a given contact 7-manifold. See Question 5.3 in Etnyre-Pancholi[6].

3.3. **Exotic contact structures on S^5.** We can insert a Lutz tube into the standard S^5. Namely,

Theorem 3.5. *In the case where $m\le 2$ and $k\neq 0$, $T_{A_{m,k}}$ is contactomorphic to the link of the isolated singular point $(0,0,0)$ of the hypersurface $\{f_{m,k}=0\}\subset\mathbb{C}^3$, where*

$$f_{1,(k_1)}=(\eta-2\xi^2)(\eta^2+2\xi^2\eta+\xi^4-\xi^{4+k_1})+\zeta^2 \quad \textit{and}$$
$$f_{2,(k_1,k_2)}=\{(\xi+\eta)^2-\xi^{2+k_1}\}\{(\xi-\eta)^2+\xi^{2+k_2}\}+\zeta^2.$$

Let $\mathcal{O}_{m,k}$ denote the Milnor fibration of the singular point $(0,0,0)\in\{f_{m,k}=0\}$.

Remark. From Theorem 2.2 and Lemma 2.5 we obtain a Reeb foliation $\mathcal{F}_{m,k}$ associated to $\mathcal{O}_{m,k}$.

In order to prove Theorem 3.5 we prepare an easy lemma.

Lemma 3.6. 1) *The complex curve*

$$C=\{\zeta^2=-(\eta-p_1)\cdots(\eta-p_{m+2})\}\quad(m=1,2)$$

on the $\eta\zeta$-plane $\mathbb{C}^2$ is topologically an m-times punctured torus in $\mathbb{R}^4$ if the points p_i are mutually distinct. These points are the critical values of the branched double covering $\pi_\eta|C$, where $\pi_\eta:\mathbb{C}^2\to\mathbb{C}$ denotes the projection to the η-axis.

2) *Let $B: p_1=p_1(\theta),\dots,p_{m+2}=p_{m+2}(\theta)$ be a closed braid on $\mathbb{C}\times S^1$ $(\theta\in S^1)$. Then the above curve $C=C_\theta$ traces a surface bundle over S^1. Fix a proper embedding $l:\mathbb{R}\to\mathbb{C}$ into the η-axis such that $l(1)=p_1(0),\dots,l(m+2)=p_{m+2}(0)$. Suppose that the closed braid B is isotopic to the geometric realization of a composition*

$$\prod_{j=1}^{J}\{\sigma_{i(j)}\}^{q(j)}\quad(q(j)\in\mathbb{Z},\ i(j)\in\{1,\dots,m+1\}),$$

where $\sigma_i:\mathbb{C}\to\mathbb{C}$ denotes the right-handed exchange of p_i and p_{i+1} along the arc $l([i,i+1])$ $(i=1,\dots,m+1)$. Then the monodromy of the surface bundle C_θ is the composition

$$\prod_{j=1}^{J}\{\tau(\ell_{i(j)})\}^{q(j)}\quad\textit{where}\quad \ell_i=(\pi_\eta|C)^{-1}(l([i,i+1])).$$

Proof of Theorem3.5. Regard $\xi\neq 0$ as a small parameter and take the branched double covering $\pi_\eta|C_\xi$ of the curve $C_\xi=\{f_{m,k}=0,\xi=\mathrm{const}\}\cap B^6$. Then the critical values of $\pi_\eta|C_\xi$ are

$$p_1,p_2=-\xi^2\{1-(\xi^{k_1})^{1/2}\}\quad\text{and}\quad p_3=2\xi_2\quad\text{in the case where}\quad m=1$$
$$(\text{resp.}\quad p_1,p_2=-\xi\{1-(\xi^{k_1})^{1/2}\},\ p_3,p_4=\xi\{1-(-\xi^{k_2})^{1/2}\}\text{ in the case where } m=2).$$

As ξ rotates along a small circle $|\xi|=\varepsilon$ once counterclockwise, the set $\{p_1,\dots,p_{m+2}\}$ traces a closed braid, which is clearly a geometric realization of the composition

$$(\sigma_1\circ\sigma_2)^6(\sigma_1)^{k_1}\qquad(\text{resp.}\quad(\sigma_1\circ\sigma_2\circ\sigma_3)^4(\sigma_1)^{k_1}(\sigma_3)^{k_2}).$$

From Lemma 3.6 and the well-known relation

$$\tau(\partial C_\xi)\simeq(\tau(\ell_1)\circ\tau(\ell_2))^6\quad(\text{resp.}\quad\tau(\partial C_\xi)\simeq(\tau(\ell_1)\circ\tau(\ell_2)\circ\tau(\ell_3))^4),$$

we see that the link of the singular point $(0,0,0) \in \{f_{m,k} = 0\}$ admits the supporting open-book decomposition in Proposition 3.2 2). This completes the proof of Theorem3.5. $\square$

If we insert the Lutz tube associated to $A_{m,k}$ $(m = 1, 2,\ k \neq 0)$ along the binding of the supporting open-book decomposition equivalent to $\mathcal{O}_{m,k}$, we obtain a contact structure $\ker(\alpha_{m,k})$ on S^5. Then the page becomes a convex overtwisted hypersurface. The following theorem can be proved in a similar way to the proof of Lemma 2.2.

Theorem 3.7. *The contact structure* $\ker(\alpha_{m,k})$ $(m = 1, 2,\ k \neq 0)$ *deforms via contact structures into a foliation which is obtained by cutting and turbulizing the page leaves of the Reeb foliation* $\mathcal{F}_{m,k}$ *along the hypersurface corresponding to the boundary of the Lutz tube.*

Let (M', α') be the contact connected sum of any contact 5-manifold (M^5, α) with the above exotic 5-sphere $(S^5, \alpha_{m,k})$. Then we see that the contact manifold $(M' \approx M^5, \alpha')$ contains a convex overtwisted hypersurface. Namely,

Theorem 3.8. *Any contact* 5*-manifold admits a contact structure which violates the Thurston-Bennequin inequality for a convex hypersurface with contact-type boundary.*

3.4. **Plastikstufes in Lutz tubes.** We show that there exists a plastikstufe in any 5-dimensional Lutz tube. First we define a plastikstufe in a contact 5-manifold.

Definition 3.9. Let (M^5, α) be a contact 5-manifold and $\iota : T^2 \to M^5$ a Legendrian embedding of the torus which extends to an embedding $\tilde{\iota} : D^2 \times S^1 \to M^5$ of the solid torus. Then the image $\tilde{\iota}(D^2 \times S^1)$ is called a *plastikstufe* if there exists a function $f(r)$ such that

$$(r^2 d\theta + f(r)dr) \wedge (\tilde{\iota}^* \alpha) \equiv 0, \quad \lim_{r \to 0} \frac{f(r)}{r^2} = 0 \quad \text{and} \quad \lim_{r \to 1} |f(r)| = \infty,$$

where r and θ denote polar coordinates on the unit disk D^2.

Example 3.10. Consider the contactization

$$(\mathbb{R} \times T_A \times \mathbb{R}(\ni t),\ \alpha = \beta_+ + s\beta_- + dt)$$

of the exact symplectic manifold $(\mathbb{R}(\ni s) \times T_A, d(\beta_+ + s\beta_-))$, where $\beta_\pm$ are the 1-forms described in Example 1.9. Take coordinates p and q near the origin on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that $p = q = 0$ at the origin, $\partial/\partial p = v_+$ and $\partial/\partial q = v_-$. Then for small $\varepsilon > 0$, the codimension-2 submanifold

$$\mathcal{P} = \{p = \varepsilon a^{-z} g(s), q = \varepsilon a^z s g(s)\} \subset \mathbb{R} \times T_A \times \mathbb{R}$$

is compactified to a plastikstufe on the Lutz tube $T_A \times D^2$, where $g(s)$ is a function with

$$g(s) \equiv 0 \quad \text{on} \quad (-\infty, 1], \quad \text{and} \quad g(s) \equiv \frac{1}{s \log s} \quad \text{on} \quad [2, \infty).$$

Note that the boundary $\{(\infty, (0, 0, z), t)) \mid z \in S^1, t \in S^1\} \approx T^2$ of the plastikstufe is a Legendrian torus, and on the submanifold $\mathcal{P}$ the contact form α can be written as

$$\alpha|\mathcal{P} = a^{-z} dq + s a^z dp + dt = \varepsilon \left(g(s) + 2sg'(s)\right) ds + dt.$$

Indeed, as $s \to \infty$,

$$g(s) \to 0, \quad sg(s) \to 0 \quad \text{and} \quad \int_2^s (g(s) + 2sg'(s)) ds \to -\infty.$$

Remark. As $\varepsilon \to 0$, the above plastikstufe converges to a solid torus $S^1 \times D^2$ foliated by S^1 times the straight rays on D^2, i.e., the leaves are $\{t = \text{const}\}$.

The following theorem is proved in the above example.

Theorem 3.11. *There exists a plastikstufe in any* 5*-dimensional Lutz tube.*

Corollary. (Niederkrüger-van Koert[23].) *Any contact* 5*-manifold* (M^5, α) *admits another contact structure* $\ker \alpha'$ *such that* (M^5, α') *contains a plastikstufe.*

3.5. **Topology of the pages.** We decide the Euler characteristic of the page of the open-book decomposition given in Theorem 3.5 which is diffeomorphic to

$$F = \{f_{m,k}(\xi, \eta, \zeta) = \delta\} \cap \{|\xi|^2 + |\eta|^2 + |\zeta|^2 \le \varepsilon\},$$

where $\delta \in \mathbb{C}, \varepsilon \in \mathbb{R}_{>0}, 0 < |\delta| \ll \varepsilon \ll 1$. Let π_ξ, π_η and π_ζ denote the projections to the axes.

In the case where $m = 1$, the critical values of $\pi_\xi|F$ are the solutions of the system

$$f_{1,(k_1)} - \delta = 0, \quad \frac{\partial}{\partial \eta} f_{1,(k_1)} = 0, \quad \frac{\partial}{\partial \zeta} f_{1,(k_1)} = 2\zeta = 0 \quad \text{and} \quad |\xi| \ll \varepsilon.$$

Therefore, for each critical value ξ of $\pi_\xi|F$, we have the factorization

$$(\eta - 2\xi^2)(\eta^2 + 2\xi^2\eta + \xi^4 - \xi^{4+k_1}) - \delta = (\eta - a)^2(\eta + 2a)$$

of the polynomial of η, where the parameter $a \in \mathbb{C}$ depends on ξ. By comparison of the coefficients of the η^1-terms and the η^0-terms we have

$$-4\xi^4 + \xi^4 - \xi^{4+k_1} = a^2 - 2a^2 - 2a^2 \quad \text{and} \quad -2\xi^6 + 2\xi^{6+k_1} - \delta = 2a^3.$$

Eliminating the parameter a, we obtain the equation

$$4\xi^{12+k_1}(9 - \xi^{k_1})^2 = 108\xi^6(1 - \xi^{k_1})\delta + 27\delta^2.$$

Then we see that $\pi_\xi|F$ has $12 + k_1$ critical points, which indeed satisfy $a \neq -2a$, i.e., the map $\pi_\xi|F$ defines a PALF structure on F with $12 + k_1$ singular fibres. Thus we have

$$\chi(F) = 1 - 2 + 12 + k_1 = 11 + k_1.$$

In the case where $m = 2$, we have the factorization

$$\{(\xi + \eta)^2 - \xi^{2+k_1}\}\{(\xi - \eta)^2 + \xi^{2+k_2}\} - \delta = (\eta - a)^2(\eta + a - b)(\eta + a + b) \quad (a, b \in \mathbb{C}).$$

By comparison of the coefficients we have

$$\begin{cases} \xi^2(2 + \xi^{k_1} - \xi^{k_2}) = 2a^2 + b^2 \\ \xi^3(\xi^{k_1} + \xi^{k_2}) = ab^2 \\ \xi^4(1 - \xi^{k_1})(1 + \xi^{k_2}) - \delta = a^2(a^2 - b^2) \end{cases}.$$

In order to eliminate a, b, we put $a = u + v$ and $\xi^2(2 + \xi^{k_1} - \xi^{k_2}) = 6uv$. Then we have

$$\begin{cases} 6uv - 2(u + v)^2 = b^2 \\ \xi^3(\xi^{k_1} + \xi^{k_2}) = -2(u^3 + v^3) \\ \xi^4(1 - \xi^{k_1})(1 + \xi^{k_2}) - \delta = (u + v)^4 + 2(u^3 + v^3)(u + v) \end{cases}.$$

Further we put

$$p = uv, \quad q = u^3 + v^3 \quad \text{and} \quad r = (u + v)^4 + 2(u^3 + v^3)(u + v).$$

Then p, q and r are polynomials of ξ. Eliminating a from

$$q(= q(p, a)) = 3pa - a^3 \quad \text{and} \quad r(= r(p, a)) = -6pa^2 + 3a^4,$$

we obtain

$$(27q^4 - r^3) + 54(prq^2 - p^3q^2) + 18p^2r^2 - 81p^4r = 0,$$

which is a polynomial equation of ξ. As $\delta \to 0$, the left hand side converges to

$$\xi^{12+k_1+k_2} \left\{ 1 - \frac{\xi^{k_1} - \xi^{k_2}}{2} + \frac{(\xi^{k_1} + \xi^{k_2})^2}{16} \right\}^2.$$

Therefore $\pi_\xi|F$ has $12 + k_1 + k_2$ critical points, which indeed satisfy $4a^2 \neq b^2$ and $b \neq 0$, i.e., the map $\pi_\xi|F$ defines a PALF structure on F with $12 + k_1 + k_2$ singular fibres. Thus we have

$$\chi(F) = 1 - 3 + 12 + k_1 + k_2 = 10 + k_1 + k_2.$$

3.6. **Symplectic proof.** In this subsection we sketch another proof of the following theorem, which is slightly weaker than Theorem 3.5 and the result of the previous subsection.

Theorem 3.12. *The contact manifold $T_{A_{m,k}}$ $(m = 1, 2)$ is contactomorphic to the binding of a supporting open-book decomposition on the standard S^5 whose page $P_{m,k}$ satisfies $\chi(P_{m,k}) = 11+k_1$ in the case where $m = 1$ and $\chi(P_{m,k}) = 10 + k_1 + k_2$ in the case where $m = 2$.*

We start with the following observation.

Observation. 1) We consider the fibre

$$V_\delta = \{\xi^2 + \eta^2 + \zeta^2 = \delta\} \subset \mathbb{C}^3$$

of the singular fibration $f(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 : \mathbb{C}^3 \to \mathbb{C}(\ni \delta)$, which we call the first fibration. If $\delta \neq 0$, the restriction $\pi_\xi|V_\delta$ is a singular fibration over the ξ-axis, which we call the second fibration. The fibre of the second fibration is

$$F_\xi = (\pi_\xi|V_\delta)^{-1}(\xi) = \{\eta^2 + \zeta^2 = \delta - \xi^2\}.$$

If $\xi^2 \neq \delta$, the restriction $\pi_\eta|F_\xi$ has critical values $\pm\gamma = (\delta - \xi^2)^{1/2}$. That is, the second fibre F_ξ is a double cover of the η-axis branched over $\pm\gamma$. We call $\pi_\eta|F_\xi$ the third fibration. Then the line segment between $\pm\gamma$ lifts to the vanishing cycle

$$\{(0, \gamma x, \gamma y) \,|\, (x, y) \in S^1 \subset \mathbb{R}^2\}(\approx S^1) \subset F_\xi$$

which shrinks to the singular points $(\delta^{1/2}, 0, 0)$ on the fibres $F_{\delta^{1/2}}$. On the other hand, the line segment between $\delta^{1/2}$ on the ξ-axis lifts to the vanishing Lagrangian sphere

$$L = \{(\delta^{1/2}x, \delta^{1/2}y, \delta^{1/2}z) \,|\, (x, y, z) \in S^2 \subset \mathbb{R}^3\}(\approx S^2) \subset V_\delta$$

which shrinks to the singular point $(0, 0, 0)$ on V_0. The monodromy of the first fibration around $\delta = 0$ is the Dehn-Seidel twist along the Lagrangian sphere L (see [10]).

2) Next we consider the *tube*

$$B' = \{|\xi + \eta^2 + \zeta^2| \leq \varepsilon\} \cap B^6$$

of the regular fibration $g(\xi, \eta, \zeta) = \xi + \eta^2 + \zeta^2 : \mathbb{C}^3 \to \mathbb{C}$. Let V'_δ denote the fibre $g^{-1}(\delta)$ $(|\delta| \leq \varepsilon)$ of the first fibration g and F'_ξ the fibre of the second fibration $\pi_\xi|V'_\delta$. The third fibration $\pi_\eta|F'_\xi$ has two critical points $(\delta - \xi)^{1/2}$ on the η-axis. Then the line segment between $(\delta - \xi)^{1/2}$ lifts to the vanishing circle

$$\{(\xi, (\delta - \xi)^{1/2}x, (\delta - \xi)^{1/2}y) \,|\, (x, y) \in S^1 \subset \mathbb{R}^2\}(\approx S^1) \subset F'_\xi$$

which shrinks to the singular point $N = (\delta, 0, 0)$ on F'_δ. By attaching a symplectic 2-handle to B', we can simultaneously add a singular fibre to each second fibration $\pi_\xi|V'_\delta$ so that the above vanishing cycle shrinks to another singular point S than N. Here the vanishing cycle traces a Lagrangian sphere from the north pole N to the south pole S. (The attaching sphere can be considered as the equator.) The symplectic handle body $B'\cup$(the 2-handle) can also be realized as a regular part

$$f^{-1}(U) \cap B^6 \quad (\exists U \approx D^2, U \not\ni 0)$$

of the singular fibration f in the above 1). Thus we can add a singular fibre $V_0 \cap B^6$ to it by attaching a symplectic 3-handle. That is, the tube $\{|f| \leq \varepsilon\} \cap B^6$ of the singular fibration f can be considered as the result of the cancellation of the 2-handle and the 3-handle attached to the above tube B' of the regular fibration g. Note that such a cancellation preserves the contactomorphism-type of the contact-type boundary.

Proof of Theorem 3.12. Take the tubes $\{|h_m| \leq \varepsilon\} \cap B^6$ $(m = 1, 2)$ of the regular fibrations

$$h_1(\xi, \eta, \zeta) = \xi + \eta^3 + \zeta^2 \quad \text{and} \quad h_2(\xi, \eta, \zeta) = \xi + \eta^4 + \zeta^2.$$

Let $F_{m,\xi}$ denote the fibre of the second fibration $\pi_\xi|h_m^{-1}(\delta)$. Then the third fibration $\pi_\eta|F_{m,\xi}$ has $m + 2$ singular fibres $(m = 1, 2)$. We connect the corresponding critical values on the η-axis by a simple arc consisting of $m + 1$ line segments $\sigma_1, \ldots, \sigma_{m+1}$, which lift to vanishing cycles

$\ell_1, \ldots, \ell_{m+1}$ on the fibre $F_{m,\xi}$. Then we can attach a symplectic 2-handle to the tube along one of the vanishing cycles $\ell_1, \ldots, \ell_{m+1}$, and cancel it by attaching a 3-handle as is described in the above observation. Iterating this procedure, we can obtain a symplectic filling of the standard S^5 as the total space of a symplectic singular fibration over D^2 whose regular fibre is an exact symplectic filling of $T_{A_{m,k}}$. Then the Euler characteristic of the fibre is $11+k_1$ ($m=1$) or $10+k_1+k_2$ ($m=2$). Indeed, the relation

$$\tau(\partial F_{1,\xi}) \simeq (\tau(\ell_1)\circ\tau(\ell_2))^6 \quad (\text{resp.} \quad \tau(\partial F_{2,\xi}) \simeq (\tau(\ell_1)\circ\tau(\ell_2)\circ\tau(\ell_3))^4)$$

implies that we can attach $12+k_1$ (resp. $12+k_1+k_2$) pairs of 2- and 3-handles to the tube $\{|h_1|\le\varepsilon\}\cap B^6$ (resp. $\{|h_2|\le\varepsilon\}\cap B^6$) to obtain the above fibration. This completes the proof of Theorem 3.12 □

Remark. 1) Giroux and Mohsen further conjectured that, for any supporting open-book decomposition on a contact manifold (M^{2n+1},α), we can attach a n-handle to the page to produce a Lagrangian n-sphere S^n, and then add a Dehn-Seidel twist along S^n to the monodromy to obtain another supporting open-book decomposition on (M^{2n+1},α) ([10]). We did a similar replacement of the supporting open-book decomposition in the above proof of Theorem 3.12 by means of symplectic handles.

2) Take a triple covering from the three-times punctured torus to the once punctured torus $F_{1,\xi}$ such that ℓ_2 lifts to a long simple closed loop. Then from the relations

$$\tau(\partial F_{1,\xi}) \simeq (\tau(\ell_1)\circ\tau(\ell_2))^6 \simeq (\tau(\ell_1)\circ\tau(\ell_2)^3)^3$$

we see that the Dehn twist along the boundary of the three-times punctured torus is also isotopic to a composition of Dehn twists along non-separating loops. On the other hand, take a double covering from the four-times punctured torus to the twice punctured torus $F_{2,\xi}$ such that ℓ_2 lifts to a long simple closed loop. Then from the relations

$$\begin{aligned}\tau(\partial F_{2,\xi}) &\simeq \{\tau(\ell_1)\}^{-1}\circ\{\tau(\ell_1)\circ\tau(\ell_3)\circ\tau(\ell_2)\}^4\circ\tau(\ell_1)\\ &\simeq \{\tau(\ell_2)\circ\tau(\ell_3)\circ\tau(\ell_2)^2\circ\tau(\ell_1)\circ\tau(\ell_2)\}^2\end{aligned}$$

we see that the Dehn twist along the boundary of the four-times punctured torus is also isotopic to a composition of Dehn twists along non-separating loops.

Problem. Can we generalize Theorem 3.12 to the case where $m=3$ or 4?

4. Further discussions

A (half) Lutz twist along a Hopf fibre in the standard S^3 produces a basic overtwisted contact manifold $\overline{S^3}$ diffeomorphic to S^3. This overtwisted contact structure is supported by the negative Hopf band. Indeed any overtwisted contact manifold $\overline{M^3}$ is a connected sum with $\overline{S^3}$ (i.e., $\overline{M^3} = \exists M^3\#\overline{S^3}$). Moreover a typical supporting open-book decomposition on $\overline{M^3}$ is the Murasugi-sum (=plumbing) with a negative Hopf band (see [10]). — The author's original motivation was to find various Lutz tubes in a given overtwisted contact 3-manifold or simply in $\overline{S^3}$. (See [20] for the first model of a Lutz tube by means of a supporting open-book decomposition.) Since the binding of the trivial supporting open-book decomposition on S^3 is a Hopf fibre, the above Lutz twist inserts a Lutz tube along the binding. Then the Lutz twist produces a non-supporting trivial open-book decomposition $\mathcal{O}$ by convex overtwisted disks. In 5-dimensional case, the Lutz tube (i.e., the neighbourhood of the binding of $\mathcal{O}$) is replaced by a 5-dimensional Lutz tube which contains a plastikstufe, and the convex overtwisted disk (i.e., the page) by a convex overtwisted hypersurface violating the Thurston-Bennequin inequality. The idea of placing a Lutz tube around the binding of a non-supporting open-book decomposition is also found in the recent work of Ishikawa[15]. However, in general, the insertion of a 5-dimensional Lutz tube requires only the normal triviality of the contact submanifold T_A in the original contact manifold.

Problem 4.1. Suppose that a contact 5-manifold (M^5,α) contains a Lutz tube. Then does it always contain a convex overtwisted hypersurface?

We also have the basic exotic contact 5-manifold $\overline{S^5}$ which is diffeomorphic to S^5 and supported by the 5-dimensional negative Hopf band. Here the negative Hopf band is the mirror image of the positive Hopf band which is (the page of) the Milnor fibration of $(0,0,0) \in \{\xi^2+\eta^2+\zeta^2=0\}$. Thus the monodromy of the negative Hopf band is the inverse of the Dehn-Seidel twist (see Obsrvation 1) in §3.6). The fundamental problem is

Problem 4.2. Does $\overline{S^5}$ contains a Lutz tube or a plastikstufe? Could it be that $\overline{S^5}$ is contactomorphic to $(S^5, \ker(\alpha_{m,k}))$? Note that almost contact structures on S^5 are mutually homotopic.

The next problem can be considered as a variation of Calabi's question (see §1).

Problem 4.3. Does the standard S^{2n+1} $(n>1)$ contains a convex hypersurface with disconnected contact-type boundary?

If there is no such hypersurfaces, the following conjecture trivially holds.

Conjecture 4.4. The Thurston-Bennequin inequality holds for any convex hypersurface with contact-type boundary in the standard S^{2n+1}.

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